How Alexander the Great Brought the Greeks Together While Inflicting Minimal Damage to the Barbarians

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Abstract

Let $\mathcal{R}$ be a finite set of red point sites in $\mathbb{R}^d$ and let $\mathcal{B}$ be a set of n blue point sites in $\mathbb{R}^d$. We want to establish "safe" connections between the red sites by deleting a minimum number of blue sites such that the region controlled by the red sites is connected. More precisely, we want to find a minimum-size subset $\mathcal{B}_{\text{del}} \subseteq \mathcal{B}$ such that the red cells in the Voronoi diagram of $\mathcal{R} \cup \mathcal{B} \setminus \mathcal{B}_{\text{del}}$ form a connected region. For $|\mathcal{R}| = 2$ we present an optimal $O(n \log n)$-time algorithm for $d = 2$, and an $O(n^{d-1})$-time algorithm for $d \geqslant 3$; we also show that the problem is 3SUM-hard for $d = 3$. Furthermore, we show that the general problem, where the number of red sites is not a constant, is NP-hard.

1 Introduction

Let $\mathcal{R}$ be a set of red cities and let $\mathcal{B}$ be a set of blue cities. Suppose each city controls a subset of the space, namely the set of all points for which it is the closest city. The red people would like to be able to travel safely between any two points in the red region, without having to cross through hostile (blue) territory. This may not always be possible, however, since the red region need not be connected. Then some blue cities will have to be eliminated, in order to make the red region connected. As the red people are a friendly people, they wish to do so by eliminating as few blue cities as possible.

In a more abstract setting, the problem above can be formulated using Voronoi diagrams: we are given a set $\mathcal{R}$ of red point sites in $\mathbb{R}^d$ and a set $\mathcal{B}$ of blue point sites in $\mathbb{R}^d$, and we want to find a subset $\mathcal{B}_{\text{del}} \subseteq \mathcal{B}$ such that the red cells in the Voronoi diagram of $\mathcal{R} \cup \mathcal{B} \setminus \mathcal{B}_{\text{del}}$ form a connected region. We call every such subset a *connectivity set*, and we want to find a connectivity set of minimum size. We call the problem of computing such a set MIN VORONOI CONNECTIVITY. We obtain the following results:

- We solve the problem for two red sites, see Section 2. Our algorithm runs in $O(n \log n + n^{d-1})$ time, where n is the number of blue sites and $d \geq 2$ is the dimension of the underlying space. We show that this is optimal for $d = 2$ and that the problem is 3SUM-hard for $d = 3$.
- We show that the general problem, where the number of red sites is not a constant, is NP-hard, see Section 3.

Terminology and notation. We denote the Voronoi diagram of a set S of sites by $\text{Vor}(S)$. We say that two sites $p, q \in S$ are *neighbors* if the boundaries of their Voronoi cells have a $(d-1)$-dimensional overlap. In other words, two sites are neighbors if they share an edge in the Delaunay graph of S.

We denote the bisector of p and q by $\beta(p, q)$. For a third site r, we call the part of $\beta(p, q)$ that lies inside or on the boundary of the Voronoi cell of r in $\text{Vor}(\{p, q, r\})$—that is, the part of $\beta(p, q)$ lying at least as close to r as to p and q—the *shadow region* of r on $\beta(p, q)$. We say that r *covers* this part of $\beta(p, q)$.

2 The Case of Two Red Sites

In this section we consider the special case that $\mathcal{R}$ consists of only two sites.

Theorem 1 *Let $\mathcal{R}$ be a set of two points in $\mathbb{R}^d$ and $\mathcal{B}$ be a set of n points in $\mathbb{R}^d$. Then* MIN VORONOI CONNECTIVITY *can be solved in $O(n \log n)$ time for $d = 2$, and in $O(n^{d-1})$ time for $d \geqslant 3$.*

Proof. Let $\mathcal{R} = \{p, q\}$, and let $\beta := \beta(p, q)$ be the bisector of p and q. For each site $b \in \mathcal{B}$, the shadow region $\sigma(b)$ on β is a $(d-1)$-dimensional half-space within the $(d-1)$-dimensional space β. Observe that p and q are neighbors in $\text{Vor}(\mathcal{B} \cup \{p, q\})$ if and only if the union of the shadow regions $\sigma(b)$ over all $b \in \mathcal{B}$ does not fully cover β. Hence, we can solve the problem as follows.

1. Let $h(b)$ denote the $(d-2)$-flat bounding $\sigma(b)$, and let $H := \{h(b) : b \in \mathcal{B}\}$. Construct the

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arrangement $\mathcal{A}(H)$ on β induced by the $(d-2)$-flats in H.

2. Compute for each cell of $\mathcal{A}(H)$ in how many shadow regions it is contained. Let C be a cell for which this number is minimum. Report the set $\mathcal{B}_{\mathrm{del}}$ of blue sites whose shadow regions cover C.

For $d \geqslant 3$, we can construct the $(d-1)$-dimensional arrangement in $O(n^{d-1})$ time [2]. For $d = 2$, we have to construct a 1-dimensional arrangement. This boils down to sorting the endpoints of the shadow regions on β, which takes $O(n \log n)$ time. After constructing the arrangement, Step 2 can be done in $O(n^{d-1})$ time, by traversing the dual graph of $\mathcal{A}(H)$ and maintaining the number of shadow regions containing the cells as we move from cell to cell in the dual graph. □

The general algorithm can be adapted so as to handle other types of sites. The time complexity again depends on the running time of the subroutine that computes the arrangement of the shadow-region boundaries on the bisector of the red sites.

Note that even in the plane, for some types of sites such as disks and ellipses, a shadow regions can consist of more than one connected component [3]. If, however, the number of connected components of each shadow region is bounded by a constant, then the number of intervals in the overlay of the blue shadow regions is linear (in the case $d = 2$). Even if β and the blue shadow regions are not of linear algebraic nature, this does not affect the time complexity of the algorithm, assuming that the necessary basic algebraic computations still take constant time.

Next we show lower bounds for Min Voronoi Connectivity.

Theorem 2 Min Voronoi Connectivity *with two red sites in* $\mathbb{R}^2$ *is in* $\Omega(n \log n)$.

Proof. Consider the problem ε-Closeness, which is defined as follows:

ε-Closeness
Input: A set $\mathcal{X}$ of n reals and a real $\varepsilon > 0$.
Output: yes if there are at least two elements in $\mathcal{X}$ whose distance is less than ε, no otherwise.

ε-Closeness has been proven to be in $\Omega(n \log n)$ for the linear decision tree model [4] and with similar arguments the same lower bound can be proven in the fixed-order algebraic decision-tree model [1]. We shall now describe a linear-time reduction from ε-Closeness to Min Voronoi Connectivity for point sites in $\mathbb{R}^2$ with $|\mathcal{R}| = 2$.

Let $(\mathcal{X}, \varepsilon)$ be an instance of ε-Closeness. We create an instance of Min Voronoi Connectivity as follows. Let $\mathcal{R} = \{(0,1),(0,-1)\}$. The bisector $\beta : y = 0$ of the two red sites represents the real axis for the instance of ε-Closeness. For each $\xi \in \mathcal{X}$ we construct two blue sites such that their shadow regions are the rays $\{(x,0) : x \leq \xi - \varepsilon\}$ and $\{(x,0) : x \geq \xi + \varepsilon\}$. Clearly, our reduction takes linear time.

The set $\mathcal{X}$ is a yes instance of ε-Closeness if and only if Min Voronoi Connectivity with input $\mathcal{R}$ and $\mathcal{B}$ has a solution that eliminates less than $n-1$ blue sites. □

Point sites in 3-space. We now prove that Min Voronoi Connectivity in $\mathbb{R}^3$ is 3sum-hard. The class of 3sum-hard problems was introduced by Gajentaan and Overmars [5]. Similar to the conjectured computational intractability of np-hard problems, 3sum-hard problems are conjectured to not allow for subquadratic algorithms (depending on the model of computation). One can show that a problem Π is 3sum-hard by giving a reduction that transforms in $o(n^2)$ time instances of a known 3sum-hard problem Π' to instances of Π.

Theorem 3 Min Voronoi Connectivity *with two red sites in* $\mathbb{R}^3$ *is* 3sum*-hard.*

Proof. We define a *strip* to be the area between two parallel lines. We consider the following problem:

Strips Cover Box
Input: A set $\mathcal{S}$ of n strips in the plane and an axis-parallel rectangle ρ.
Output: yes if the union of the strips completely covers the area of ρ, no otherwise.

Strips Cover Box is 3sum-hard [5]. We give a linear-time reduction from Strips Cover Box to Min Voronoi Connectivity for point sites in $\mathbb{R}^3$ with $|\mathcal{R}| = 2$.

Let $(\mathcal{S}, \rho)$ be an instance of Strips Cover Box. We create an instance of Min Voronoi Connectivity as follows. Let $\mathcal{R} = \{(0,0,1),(0,0,-1)\}$. The bisector $\beta : z = 0$ of the two red sites represents the plane that contains ρ and the strips in $\mathcal{S}$.

For each edge e of ρ, we construct $n+1$ blue sites such that their shadow regions are identical, having a boundary on β that coincides with the support line of e, and they do not contain ρ. We can make multiple sites have the same shadow region by placing them on the perimeter of a circle that lies on a plane orthogonal to β. Thus, every point of β outside ρ is covered by at least $n+1$ sites. For each strip in $\mathcal{S}$ we construct two blue sites such that the intersection of their shadow regions on β coincides with the strip. Clearly, our reduction takes linear time.

The instance $(\mathcal{S}, \rho)$ is a yes-instance for Strips Cover Box if and only if any solution to Min Voronoi Connectivity with input $\mathcal{R}$ and $\mathcal{B}$ eliminates at least $n+1$ blue sites. This shows that in 3-space Min Voronoi Connectivity is 3sum-hard. □

3 The General Case

We now investigate the complexity of the decision version of MIN VORONOI CONNECTIVITY for point sites in $\mathbb{R}^2$ where we drop the restriction that $|\mathcal{R}| = 2$. In other words, we consider the following problem:

VORONOI CONNECTIVITY
Input: Two sets $\mathcal{R}$ and $\mathcal{B}$ of point sites in the plane and a natural number k.
Output: YES if there exists a connectivity set $\mathcal{B}_{\text{del}} \subseteq \mathcal{B}$ with $|\mathcal{B}_{\text{del}}| \leqslant k$, NO otherwise.

For a given undirected graph $G = (V, E)$ a set of vertices $C \subseteq V$ is a *connected vertex cover* if the subgraph induced by C is connected and C is a vertex cover of G, that is, C contains at least one endpoint of each edge of G. To show that VORONOI CONNECTIVITY is NP-hard, we reduce from the following special case of connected vertex cover, which is NP-hard [6].

PLANAR CONNECTED VERTEX COVER
Input: A planar 2-connected graph $G = (V, E)$ of maximum degree 4 and a positive integer k.
Output: YES if there exists a connected vertex cover of G that consists of at most k vertices, NO otherwise.

The reduction. Let (G, k) be an instance of PLANAR CONNECTED VERTEX COVER. Our approach is as follows. We first construct a rectilinear embedding of G on a grid of size polynomial in $n = |V|$. Then we use the grid coordinates of the vertices of G to place the red and blue sites. We prove that in the induced Voronoi diagram we can connect the Voronoi cells of the red sites by deleting at most k blue sites if and only if G has a connected vertex cover of size at most k.

First we compute a planar grid embedding of G using the algorithm of Tamassia and Tollis [7]. Their linear-time algorithm maps the vertices of G to distinct points of an $O(n^2)$-size section of the integer grid and maps the edges of G to non-intersecting rectilinear paths over grid points. To any grid point $p = (x_p, y_p)$ that appears in this embedding, we assign the square $[x_p - 1/2, x_p + 1/2] \times [y_p - 1/2, y_p + 1/2]$. Let $E_1(G)$ denote the set of grid squares occupied by the resulting embedding of G. We subdivide each square $\delta \in E_1(G)$ into a grid of $(2n + 1) \times (2n + 1)$ squares that we denote by *grid*(δ). We denote the set of squares of the latter, refined embedding of G by $E_2(G)$. We place either a red or a blue site in the center of each square of *grid*(δ). We call such a square a red or blue square according to the color of the site that we placed in its center. We place the red sites so that the red squares in $E_2(G)$ form a rectilinear embedding of G very similar to $E_1(G)$ yet now the vertices and the rectilinear edge paths have the thickness of one square of $E_2(G)$. In Figure 1 we show the patterns in which we place red and blue sites in the grid squares of *grid*(δ) for each $\delta \in E_1(G)$. If δ corresponds to a vertex in G, then we place a blue site in the center square of *grid*(δ). We call this site a *vertex site*. If δ does not correspond to a vertex, we place a red site in this square. In each grid square in $E_2(G)$ that is not occupied by a red site or a vertex site, we place a blue site. Hence, on each side of a rectilinear path of red squares, there is a "padding" of n disjoint rectilinear paths of blue squares.

Note that in the induced Voronoi diagram no two red neighbors of a vertex site share a common boundary edge of their cells.

We denote the sets of red and blue sites that we place into the squares of $E_2(G)$ by $\mathcal{R}(G)$ and $\mathcal{B}(G)$, respectively. Note that in the Voronoi diagram induced by $\mathcal{R}(G)$ and $\mathcal{B}(G)$, the Voronoi cells of the red sites form one connected component for each embedded edge of G. We call these components *edge components*. We can construct $\mathcal{R}(G)$ and $\mathcal{B}(G)$ in time polynomial in n.

Lemma 4 *The Voronoi diagram of $\mathcal{R}(G)$ and $\mathcal{B}(G)$ has a connectivity set of size at most k if and only if G admits a connected vertex cover of size at most k.*

Proof. "if": Let C be a connected vertex cover of G, and let $\mathcal{B}_{\text{del}} \subseteq \mathcal{B}(G)$ be the set of vertex sites that correspond to the vertices in C. Clearly, $|C| = |\mathcal{B}_{\text{del}}|$.

We argue that $\mathcal{B}_{\text{del}}$ is a connectivity set for $\text{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G))$. Take any two red sites $s \neq s'$. Each of these lies on the embedding of an edge of G. Since C is a vertex cover, each of the two edges has an endpoint in C. Since C is a connected vertex cover, the graph induced by C contains a path between the two endpoints. Consider the set of sites that lie between s and s' on the embedding of this path plus the two initial edges. All blue sites in this set lie in $\mathcal{B}_{\text{del}}$. Removing $\mathcal{B}_{\text{del}}$ from $\mathcal{B}(G)$ connects the Voronoi cells of all red sites in the set, in particular, those of s and s'. Hence $\mathcal{B}_{\text{del}}$ is a connectivity set.

"only if": Let $\mathcal{B}_{\text{del}} \subseteq \mathcal{B}(G)$ be a connectivity set for $\text{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G))$. We first assume that all sites in $\mathcal{B}_{\text{del}}$ are vertex sites. Let C be the set of vertices in G that correspond to sites in $\mathcal{B}_{\text{del}}$. Then C is a vertex cover of G. Otherwise there is at least one edge component in $\text{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G) \setminus \mathcal{B}_{\text{del}})$ that has not merged with any of the other components. The graph induced by C must also be connected—otherwise the red cells in $\text{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G) \setminus \mathcal{B}_{\text{del}})$ form more than one connected component.

It remains to examine the case that $\mathcal{B}_{\text{del}}$ contains also padding sites. We show that there is a connectivity set $\mathcal{B}'_{\text{del}} \subseteq \mathcal{B}(G)$ with $|\mathcal{B}'_{\text{del}}| \leq |\mathcal{B}_{\text{del}}|$ that contains more vertex sites than $\mathcal{B}_{\text{del}}$. (We can repeat this argument until we have only vertex sites.) Let c and c' be two distinct edge components in $\text{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G))$.

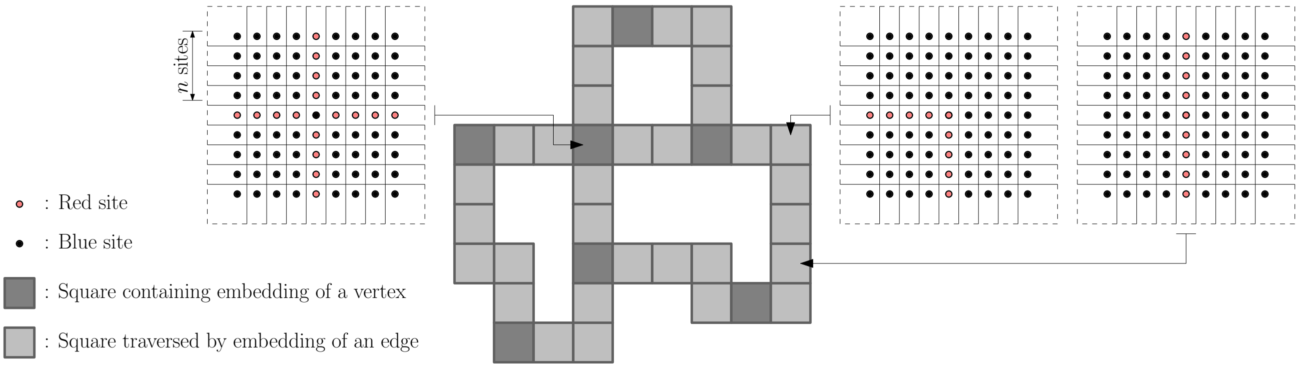

Figure 1: The coarser grid embedding $E_1(G)$ (gray shaded squares) along with the induced Voronoi diagram of the three main possible patterns for $grid(\delta)$ in the reduction of VORONOI CONNECTIVITY. The dark gray squares correspond to the vertices of G.

Let s be a site whose cell belongs to c, and let s' be a site whose cell belongs to c'. We consider two cases.

First, suppose the cells of s and s' are adjacent in $\mathrm{Vor}(\mathcal{R}(G) \cup \mathcal{B}(G) \setminus \mathcal{B}_{\mathrm{del}})$. If there is no vertex site whose cell is incident to both c and c' then it is easy to see that at least $2n$ blue sites must be deleted so that the cells of s and s' become adjacent. Thus $|\mathcal{B}_{\mathrm{del}}| \geqslant 2n$, and we simply let $\mathcal{B}'_{\mathrm{del}}$ be the set of all n vertex sites.

Now suppose c and c' are both incident to the square $\delta \in E_1(G)$ of some vertex site s''. If the squares in $E_2(G)$ occupied by s and s' both lie in $grid(\delta)$ then there is at least one blue site s''' in $\mathcal{B}_{\mathrm{del}}$ whose square lies in $grid(\delta)$, too. In that case we let $\mathcal{B}'_{\mathrm{del}} = (\mathcal{B}_{\mathrm{del}} \setminus \{s'''\}) \cup \{s''\}$. If at least one of the squares of s and s' in $E_2(G)$ does not lie in $grid(\delta)$ then again $|\mathcal{B}_{\mathrm{del}}| \geqslant 2n$, and we let $\mathcal{B}'_{\mathrm{del}}$ be the set of all vertex sites. In each case, we have a new connectivity set $\mathcal{B}'_{\mathrm{del}}$ with more vertex sites than $\mathcal{B}_{\mathrm{del}}$ and with $|\mathcal{B}'_{\mathrm{del}}| \leq |\mathcal{B}_{\mathrm{del}}|$. $\square$

We have just proved that VORONOI CONNECTIVITY is NP-hard since PLANAR CONNECTED VERTEX COVER is NP-hard. VORONOI CONNECTIVITY is in NP since, given $\mathcal{R}$ and $\mathcal{B}$, we can guess a potential solution $\mathcal{B}_{\mathrm{del}}$ with positive probability and then check in time polynomial in $|\mathcal{R}| + |\mathcal{B}|$ whether $\mathcal{B}_{\mathrm{del}}$ is indeed a solution by computing $\mathrm{Vor}(\mathcal{R} \cup \mathcal{B} \setminus \mathcal{B}_{\mathrm{del}})$.

Theorem 5 VORONOI CONNECTIVITY *is* NP*-complete.*

Remark 6 The proof of Theorem 5 also holds if the sites in $\mathcal{B}(G)$ and $\mathcal{R}(G)$ are perturbed slightly away from the centers of the squares in $E_2(G)$. Hence, the theorem is also applicable for non-degenerate distributions of the input sites.

4 Concluding remarks

We have introduced the problem MIN VORONOI CONNECTIVITY, and shown how it can be solved in $O(n \log n + n^{d-1})$ time for two red sites and n blue sites in $\mathbb{R}^d$. The running time of our algorithm is optimal for $d = 2$, and also for $d = 3$ if the conjectured lower bound for 3SUM holds. Our algorithm can also be used to compute all connectivity sets which are minimal under inclusion. This allows us to find an optimal connectivity set for any constant number of red sites: for every spanning tree of the complete graph on the red sites, try every combination of inclusion-minimal connectivity sets for the edges. For a non-constant number of red sites, the problem is NP-hard. $O(|\mathcal{B}|)$- and $O(|\mathcal{R}|)$-approximations are not difficult, but we haven't come up with an $O(1)$-approximation.

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